

A note on the product homomorphism problem and CQ-definability*

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The *product homomorphism problem* (PHP) takes as input a finite collection of relational structures $\mathbf{A}_1, \dots, \mathbf{A}_n$ and another relational structure \mathbf{B} , all over the same schema, and asks whether there is a homomorphism from the direct product $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$ to \mathbf{B} . This problem is clearly solvable in non-deterministic exponential time. It follows from results in [1] that the problem is NEXPTIME-complete. The proof, based on a reduction from an exponential tiling problem, uses structures of bounded domain size but with relations of unbounded arity. In this note, we provide a self-contained proof of NEXPTIME-hardness of PHP, and we show that it holds already for directed graphs, as well as for structures of bounded arity with a bounded domain size (but without a bound on the number of relations). More precisely, we obtain:

Theorem 1. *The PHP is NEXPTIME-complete [1]. The lower bound holds already for*

1. *structures with binary relations and a bounded domain size;*
2. *structures with a single relation and a bounded domain size;*
3. *structures with a single binary relation*

This completes the picture, since PHP is solvable in polynomial time when all three of the above parameters (i.e., number of relations, arity, and domain size) are bounded, as follows from the fact that, in this case, there are only finitely many different possible input structures up to isomorphism.

Theorem 1.1 is proved by an adaptations of the technique used in [1]. Theorem 1.2 is proved by a reduction from 1.1. Theorem 1.3 is proved by a reduction from Theorem 1.2.

We also present an application of the above result to the CQ-definability problem (also known as the PP-definability problem).

1 Proof of Theorem 1.1

Proof. By reduction from the exponential tiling problem. We are assuming a fixed set of tile types with associated horizontal and vertical compatibility relations, and the input of the tiling problem consists of an integer m (specified in unary) together with a sequence of (not necessarily distinct) tile types t_1, \dots, t_m . The problem is to decide whether the 2^m -by- 2^m grid has a valid tiling where t_1, \dots, t_m is a prefix of the sequence of tiles on the first row, starting at the origin. It is known that there is a fixed finite set of tile types for which this problem is NEXPTIME-hard.

For ease of exposition, we will also make use of unary relations. These can easily be replaced by binary ones.

The idea of the reduction is very simple. We will define $2m$ structures, $\mathbf{A}_1, \dots, \mathbf{A}_{2m}$, each having domain $\{0, 1\}$. In this way, each element of the product $\prod_i \mathbf{A}_i$ is a bitstring of length $2m$, which we will interpret as a pair of bistrings of length m , where the first bitstring is the binary encoding of a

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horizontal coordinate and the second bitstring is the binary encoding of a vertical coordinate. The structure \mathbf{B} will have one element for each tile type, so that a map $h : \Pi_i \mathbf{A}_i \rightarrow \mathbf{B}$ can be viewed as a way of assigning a tile type to each position on the 2^m -by- 2^m grid. Furthermore, by endowing the structures involved with suitable relations, we will ensure that every homomorphism $h : \Pi_i \mathbf{A}_i \rightarrow \mathbf{B}$ corresponds to a valid tiling, and vice versa.

Let $H, V \subseteq (\{0, 1\}^{2m})^2$ be the horizontal and vertical successor relations on coordinate pairs. In other words, $H = \{(\mathbf{x}\mathbf{y}, \mathbf{x}'\mathbf{y}') \mid \mathbf{x}' = \mathbf{x} + 1\}$ and $V = \{(\mathbf{x}\mathbf{y}, \mathbf{x}'\mathbf{y}') \mid \mathbf{y}' = \mathbf{y} + 1\}$. Let P_0, \dots, P_m be singleton sets denoting the coordinate pairs $(0, 0), \dots, (m-1, 0)$. In order to make our reduction work, we need to somehow make sure that the relations H, V, P_0, \dots, P_m are “available” in the product structure $\Pi_i \mathbf{A}_i$, by choosing the factor structures $\mathbf{A}_1, \dots, \mathbf{A}_{2m}$ appropriately.

Let us say that an n -ary relation R over the domain $\{0, 1\}^{2m}$ is *decomposable* if it can be represented as a product $R_1 \times \dots \times R_{2m}$ where each R_i is an n -ary relation over $\{0, 1\}$. Intuitively, this means that if we include in each factor structure \mathbf{A}_i the relation R_i , then the product structure $\Pi_i \mathbf{A}_i$ contains the relation R . Each of the unary relations P_0, \dots, P_m , being a singleton, is trivially decomposable. Indeed, if we use the notation $\underline{k}[i]$ to denote the value of the i -th bit of the binary encoding of the number k , then $P_k = \{\underline{k}[1]\} \times \dots \times \{\underline{k}[m]\} \times \{0\}^m$. The binary relations H is *not* decomposable. However, it turns out to be a union of decomposable relations, which will suffice for our purposes. First, observe that whenever $(\mathbf{x}\mathbf{y}, \mathbf{x}'\mathbf{y}') \in H$ then $\mathbf{x}' = \mathbf{x} + 1$, and therefore \mathbf{x} must have at least one bit that is set to zero. For each $k \leq m$, let H_k be the subrelation of H containing all $(\mathbf{x}\mathbf{y}, \mathbf{x}'\mathbf{y}') \in H$ for which it is the case that the k -th bit of \mathbf{x} is the least significant bit that is 0. By definition, we have that $H = \bigcup_k H_k$. Then H_k decomposes: $H_k = \text{id}^{k-1} \times \text{diff}^{m-k+1} \times \text{id}^m$, where id is the identity relation on $\{0, 1\}$ and diff is the difference relation on $\{0, 1\}$. The exact same story holds for V (where we have that $V_k = \text{id}^m \times \text{id}^{k-1} \times \text{diff}^{m-k+1}$).

We are now ready to defined the structures $\mathbf{A}_1, \dots, \mathbf{A}_{2m}$ and \mathbf{B} . The signature consists of the relations $H_1, \dots, H_m, V_1, \dots, V_m, P_0, \dots, P_m$. For $m, \ell < k$, we define

$$\begin{aligned} P_k^{\mathbf{A}_\ell} &= \begin{cases} \{\underline{k}[\ell]\} & \text{if } \ell \leq m \\ \{0\} & \text{otherwise} \end{cases} \\ H_k^{\mathbf{A}_\ell} &= \begin{cases} \text{diff} & \text{if } \ell \in [k, m] \\ \text{id} & \text{otherwise} \end{cases} \\ V_k^{\mathbf{A}_\ell} &= \begin{cases} \text{diff} & \text{if } \ell \in [m+k, 2m] \\ \text{id} & \text{otherwise} \end{cases} \end{aligned}$$

The structure \mathbf{B} is defined as follows: its domain is the set of all tile types. The unary predicate P_i denotes the singleton set $\{t_i\}$ as specified in the instance of the tiling problem. The relations H_k and V_k contain all pairs of tile types that are horizontally, respectively vertically, compatible.

It is now straightforward to verify that there is a homomorphism $h : \Pi_i \mathbf{A}_i \rightarrow \mathbf{B}$ if and only if there is a valid tiling. \square

2 Proof of Theorem 1.2

Proof. The proof proceed by a reduction from Theorem 1.1, which states that PHP is in NEXPTIME even for structures of a bounded domain size. The reduction goes in two steps. We first reduce to the case with two relations. Let \mathbf{B} be any structure with domain D and with multiple relations R_1, \dots, R_k of respective arity r_1, \dots, r_k over D . We denote by \mathbf{B}^* the structure with domain $D \cup \{0\}$ that has

- (i) a unary relation P denoting the set D
- (ii) a relation R of arity $r_1 + \dots + r_k$ consisting of the all-zeroes tuple $(0, \dots, 0)$, and, for every $(a_1 \dots a_{r_i}) \in R_i$ ($1 \leq i \leq k$), the tuple whose first $r_1 + \dots + r_{i-1}$ coordinates are all 0, whose subsequent r_i coordinates are $a_1 \dots a_{r_i}$, and whose final $r_{i+1} + \dots + r_k$ coordinates are 0 again.

This transformation can be carried out in polynomial time, and it increases the domain of each structure with at most one element. Furthermore, we claim that $\Pi_i \mathbf{A}_i^* \rightarrow \mathbf{B}^*$ if and only if $\Pi_i \mathbf{A}_i \rightarrow \mathbf{B}$. In one direction, suppose $h : \Pi_i \mathbf{A}_i^* \rightarrow \mathbf{B}^*$. By construction (and, more specifically, due to the presence of the unary relation P), h must map every element of $\Pi_i \mathbf{A}_i$ to an element of \mathbf{B} . It is then easy to see that h is in fact a homomorphism from $\Pi_i \mathbf{A}_i$ to \mathbf{B} . Conversely, suppose $h : \Pi_i \mathbf{A}_i \rightarrow \mathbf{B}$. Let h' be the map from $\Pi_i \mathbf{A}_i^*$ to \mathbf{B}^* that extends h such that every element of $\Pi_i \mathbf{A}_i^*$ containing a 0 is sent to the element 0 of \mathbf{B}^* . Then h' is a homomorphism from $\Pi_i \mathbf{A}_i^*$ to \mathbf{B}^* . This follows from the fact that (i) no element containing a 0 can belong to the P relation in $\Pi_i \mathbf{A}_i^*$, and (ii) if a tuple in the relation R of $\Pi_i \mathbf{A}_i^*$ includes an element containing a 0, then this tuple consists entirely of elements that contain a 0, and hence h' maps the tuple in question to the all-zeroes tuple, which belongs again to R in \mathbf{B}^* .

As a final step, we further reduce to the case with a single relation. This is done by replacing each structure with two relations, P and R , by the structure with the same domain and with a single relation that is defined as the cartesian product of P and R . Again, this transformation can be carried out in polynomial time, it does not affect the domains of the structures involved, and it preserves the existence or non-existence of a homomorphism from $\Pi_i \mathbf{A}_i^*$ to \mathbf{B}^* . \square

3 Proof of Theorem 1.3

Proof. We shall give a reduction from the PHP with a single relation (Theorem 1.2). Let $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{B}$ be an instance of the PHP, using a single r -ary relation R . We may assume without loss of generality that, for each structure $\mathbf{C} = (C, R^C)$ among $\mathbf{A}_1, \dots, \mathbf{A}_n, \mathbf{B}$, the projection of R^C to the first coordinate is the entire domain C . This is because we can always replace the r -ary relation R by the $r+1$ -ary relation $C \times R$ where \times indicates here the cartesian product. This transformation can be carried out in polynomial time and it does not affect the existence or non-existence of a homomorphism from $\Pi_i \mathbf{A}_i$ to \mathbf{B} . Henceforth, we shall use R to denote the unique relation, and r to denote its arity. If t is a r -ary tuple and $j \in \{1, \dots, r\}$ we shall denote by $t[j]$ the j th component of t .

For every i , we define $G(\mathbf{A}_i)$ to be the following digraph:

The nodes of $G(\mathbf{A}_i)$ include all elements of \mathbf{A}_i . Furthermore, for every tuple $t = (a_1, \dots, a_r) \in R_i$, $G(\mathbf{A}_i)$ contains r additional nodes, which we denote by t^j with $j = 1, \dots, r$. These nodes are connected by the following directed edges:

- (t^j, t^{j+1}) for every $1 \leq j < r$.
- $(t[j], t^j)$ for every $1 \leq j \leq r$.

We define $G(\mathbf{B})$ as the digraph obtained from \mathbf{B} in the same way, except that we further add $r-1$ additional elements s^1, \dots, s^{r-1} called *sink nodes*, connected by edges (s^j, s^{j+1}) for every $1 \leq j < r-1$, and an edge from every element of \mathbf{B} to every sink node.

Claim: there is a homomorphism $h : \Pi_i G(\mathbf{A}_i) \rightarrow G(\mathbf{B})$ if and only if there is a homomorphism $h' : \Pi_i \mathbf{A}_i \rightarrow \mathbf{B}$.

In the remainder, we prove this claim, which immediately implies the theorem. We start with the more difficult direction: let h be a homomorphism from $\Pi_i \mathbf{A}_i$ to \mathbf{B} . We shall define from h a homomorphism h' from $\Pi_i G(\mathbf{A}_i)$ to $G(\mathbf{B})$. Let $v = (v_1, \dots, v_n)$ be a node of $\Pi_i G(\mathbf{A}_i)$.

- If $v_i \in \mathbf{A}_i$ for all i then we say that v is of “type 1”. In this case we define $h'(v) = h(v)$.
- If, for all i , $v_i = t_i^{j_i}$ where t_i is a tuple in (the relation of) \mathbf{A}_i and $j_i \in \{1, \dots, r\}$ then:
 - If, in addition, there exists some j such that $j_i = j$ for every i then we say that v is of “type 2”. Note that $t_1 \times \dots \times t_m$ is a tuple in $\Pi_i \mathbf{A}_i$ and hence $h(t_1 \times \dots \times t_m)$ (where h is applied component-wise) is a tuple of \mathbf{B} . In this case, define $h'(v)$ to be $h(t_1 \times \dots \times t_m)^j$.

- Otherwise we say that v is of “type 3” and we set $h(v')$ to the sink node s^j where $j = \min j_i$. Observe that, in this case, necessarily $j \leq r - 1$.

- If v is not in any of the previous types then we say that is of “type 4”. In this case, we shall prove there exists a vertex u of type 1 such that for every vertex w of type 2 the following holds:

$$(v, w) \text{ is an edge of } \Pi_i G(\mathbf{A}_i) \Rightarrow (u, w) \text{ is an edge of } \Pi_i G(\mathbf{A}_i)$$

In this case we set $h'(v) = h'(u)$. Let us show that such u exists. If there exists i, i' such that $v_i = t_i^{j_i}$ and $v_{i'} = t_{i'}^{j_{i'}}$ and $j_i \neq j_{i'}$ then clearly v does not have an outgoing edge to any vertex of type 2 and we can set u to be any arbitrary vertex of type 1. Same applies if there exist i such that $v_i = t_i^r$. Consequently we are left with the case in which there exists some $j \in \{1, \dots, r-1\}$ such that for every i , $v_i \in \mathbf{A}_i$ or $v_i = t_i^j$ for some tuple t_i in \mathbf{A}_i . Define u_i to be v_i in the first case and $t_i[j+1]$ in the second and set $u = (u_1, \dots, u_m)$.

Let $w = (w_1, \dots, w_n)$ be a node of type 2. We shall prove that for every i , if (v_i, w_i) is an edge of $\Pi_i G(\mathbf{A}_i)$ then so if (u_i, w_i) . The claim is obvious whenever $u_i = v_i$. Assume now that $v_i = t_i^j$. Since t_i^j has only one outgoing edge (to t_i^{j+1}) in $G(\mathbf{A}_i)$ it follows that $w_i = t_i^{j+1}$. The claim follows from the fact that $u_i = t_i[j+1]$ and $G(\mathbf{A}_i)$ contains edge $(t_i[j+1], t_i^{j+1})$.

Let us prove that h' is indeed a homomorphism. Let (u, v) be an edge in $\Pi_i G(\mathbf{A}_i)$ and let $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$. We shall prove that $(h(u), h(v))$ belongs to $G(\mathbf{B})$ by means of a case analysis on the types of u and v . Notice that v is necessarily of type 2 or 3 since nodes of type 1 or 4 do not have incoming edges.

- u is of type 1. If v is of type 3 the claim follows from the fact that $G(\mathbf{B})$ has an edge from every element in \mathbf{B} to every sink vertex. Assume now that v is of type 2, that is, of the form (t_1^j, \dots, t_m^j) . Since (u, v) is an edge of $\Pi_i G(\mathbf{A}_i)$ and u is of type 1 it follows that $u_i = t_i[j]$ for every i . Hence $u = (t_1 \times \dots \times t_m)[j]$ and, since h defines a homomorphism, $h(u)$ is the j th component of $h(t_1 \times \dots \times t_m)$ (h is applied component-wise). It follows that $G(\mathbf{B})$ contains the edge from $h'(u)$ to $h'(v) = h(t_1 \times \dots \times t_m)^j$.
- u is of type 2. Then necessarily there exists t_1, \dots, t_m and j such that $u = (t_1^j, \dots, t_m^j)$ and $v = (t_1^{j+1}, \dots, t_m^{j+1})$ and the claim follows directly from the definitions.
- u is of type 3 then v is necessarily of type 3 as well. Furthermore, it follows that if $h'(u)$ is s^j then necessarily $h'(v) = s^{j+1}$.
- u is of type 4. It follows directly from the definition of $h'(u)$ and the fact that every vertex of type 3 is mapped by h' to a sink node.

Conversely, let h' be a homomorphism from $\Pi_i G(\mathbf{A}_i)$ to $G(\mathbf{B})$. Recall that each element of $\Pi_i \mathbf{A}_i$ is in particular an element of $\Pi_i G(\mathbf{A}_i)$. We claim that the restriction of h' to $\Pi_i \mathbf{A}_i$ is a homomorphism from $\Pi_i \mathbf{A}_i$ to \mathbf{B} .

First, we show that, for each element t of $\Pi_i \mathbf{A}_i$, $h'(t)$ is an element of \mathbf{B} . Let $t = (t_1, \dots, t_n)$ be any element of $\Pi_i \mathbf{A}_i$. Recall that we have assumed that the projection of $R^{\mathbf{A}_i}$ on the first coordinate is the entire domain of \mathbf{A}_i . Hence, each t_i is the first component of some tuple in $R^{\mathbf{A}_i}$. By construction of $G(\mathbf{A}_i)$, this implies that t_i has an outgoing path of length r in $G(\mathbf{A}_i)$, and hence, t has an outgoing path of length r in $\Pi_i \mathbf{A}_i$. It follows that h' must map t to a node of $G(\mathbf{B})$ that has an outgoing path of length r . By construction of $G(\mathbf{B})$, then, $h(t)$ must be an element of \mathbf{B} .

Next, we shall show that $h : \Pi_i \mathbf{A}_i \rightarrow \mathbf{B}$ is a homomorphism. Let $(t^1, \dots, t^r) \in R^{\Pi_i \mathbf{A}_i}$, where each $t^j = (t^j[1], \dots, t^j[n])$. Then we have that $(t^1[i], \dots, t^r[i])$ belongs to $R^{\mathbf{A}_i}$, for each $i \leq n$. Consequently, $(t^1[i], \dots, t^r[i])$ satisfies the conjunctive query

$$q(x_1, \dots, x_r) = \exists y_1 \dots y_r \left(\bigwedge_{1 \leq i \leq r} E(x_i, y_i) \wedge \bigwedge_{1 \leq i < r} E(y_i, y_{i+1}) \right)$$

It follows that (t^1, \dots, t^r) satisfies the same conjunctive query in $\Pi_i G(\mathbf{A}_i)$, and therefore, since conjunctive queries are preserved by homomorphisms, $h(t^1, \dots, t^r)$ satisfies q in $G(\mathbf{B})$. It follows by construction of $G(\mathbf{B})$ that $h(t^1, \dots, t^r) \in R^{\mathbf{B}}$. \square

4 Application: CQ-definability

The *CQ-definability problem* (also known under the name PP-definability, and several other names), is the problem with input an instance I and a relation S over the domain of I , to decide whether there is a conjunctive query q such that $q(I) = S$. It has been long known that this problem is decidable in coNEXPTIME (see discussion and references in [1]). It was shown in [1] that the CQ-definability problem is coNEXPTIME -complete, even for instances of a bounded domain size. On the other hand, the proof used relations of arbitrarily large arity. We show that the same problem is coNEXPTIME -complete for a fixed schema (but without a bound on the size of the domains of the instances).

Theorem 2. *The CQ-definability problem is coNEXPTIME -hard already for unary queries over a fixed schema consisting of a single binary relation.*

Proof. Reduction from PHP with a single binary relation R (Theorem 1.3). Let instances $\mathbf{A}_1, \dots, \mathbf{A}_n$ and \mathbf{B} be given. Inspection of the proof of Theorem 1.3 shows that we may assume that, in each of these structures, the maximum length of a directed path is precisely r , for some fixed natural number r . Let \mathbf{C} be the instance consisting of the disjoint union of $\mathbf{A}_1, \dots, \mathbf{A}_n$ and \mathbf{B} , extended with the facts $R(a_i, x)$ for all $i \leq n$ and $x \in \mathbf{A}_i$, and $R(b, x)$ for all $x \in \mathbf{B}$, where a_1, \dots, a_n and b are fresh elements. Observe that each a_i , and also b , by construction, has an outgoing path of length $r + 1$, while no other elements have an outgoing path of length $r + 1$. Let $S = \{a_1, \dots, a_n\}$. Then we claim that $\mathbf{A}_1 \times \dots \times \mathbf{A}_n \rightarrow \mathbf{B}$ if and only if S is not definable inside \mathbf{C} by a conjunctive query. In one direction, if $\mathbf{A}_1 \times \dots \times \mathbf{A}_n \rightarrow \mathbf{B}$ then clearly S is not definable by a conjunctive query, because, by homomorphism preservation, the same conjunctive query would have to select b . On the other hand, if $\mathbf{A}_1 \times \dots \times \mathbf{A}_n \not\rightarrow \mathbf{B}$, then we can construct a query q defining S as follows: first we define q_1 to be the canonical Boolean conjunctive query of $\mathbf{A}_1 \times \dots \times \mathbf{A}_n$, and we define $q(x)$ to be the unary conjunctive query expressing that q_1 holds in the submodel of \mathbf{C} consisting of all elements reachable (in one step) from x . By construction, $q(\mathbf{A})$ includes all of S and excludes b . It is also easy to see that $q(\mathbf{A})$ contains no elements other than a_1, \dots, a_n and b . Therefore, q defines S . \square

References

- [1] Ross Willard. Testing expressibility is hard. In David Cohen, editor, *CP*, volume 6308 of *Lecture Notes in Computer Science*, pages 9–23. Springer, 2010.